

Dynamic Characteristics of Membrane Ions in Multifield Configurations of Low-Frequency Electromagnetic Radiation

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We seek to extend the recent suggestion that classical cyclotron resonance of biologically important ions is implicated in weak electromagnetic field-cell interactions. The motion of charged particles in a constant magnetic field and periodic electric field is examined under the simplifying assumption of no damping. Each of the nine terms of the relative dielectric tensor is found to have a dependence on functions that include the factor $(\omega^2 - \omega_B^2)^{-1}$, where ω_B is the gyrofrequency. We also find a plasmalike decomposition of the electric field into oppositely rotating components that could conceivably act to drive oppositely charged ions in the same direction through helical membrane channels. For weak low-frequency magnetic fields, an additional feature arises, namely, periodic reinforcement of the resonance condition with intervals of the order of tens of msec for biological ions such as Li^+ , Na^+ , and K^+ .

Key words: cyclotron resonance, ion channels, weak electromagnetic field effects, ELF bioeffects

INTRODUCTION

A new model has been proposed [Liboff, 1985] for the interaction of low-frequency magnetic fields and cells in which biologically important ions such as Li^+ , K^+ , and Na^+ are accelerated through the membrane as a result of a cyclotron resonance mechanism determined by the frequency of the applied field and the magnitude of the local magnetostatic field (eg, the geomagnetic field). In part, this suggestion was made to explain recent data taken by Blackman et al [1985] showing that specific combinations of static and periodic magnetic fields appear to enhance Ca^{2+} -efflux from chick brain in vitro. One interesting aspect of the model uses proposed helical membrane channels that not only act to sustain the cyclotron resonance condition but also serve as low-noise regions in which the collision frequency is less than the gyrofrequency.

In the following, we provide a brief introduction to the basic theory governing the interactions between free ions and multifields (ie, mixed magnetic and electric fields and/or mixed static and periodic fields) and probe these interactions for possible cyclotron resonance effects in biostructures.

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THEORY

The basis for charged particle motion in combined electric and magnetic fields is the force equation

$$\vec{F} = m \frac{d\vec{u}}{dt} = q\vec{E} + q(\vec{u} \times \vec{B}), \quad (1)$$

where $\vec{F}$ is the force vector in Newtons, m is the mass of the charged particle in kg, q is the charge on the particle in Coulombs, $\vec{E}$ is the electric field intensity in V/m, $\vec{B}$ is the magnetic flux density in Tesla, and $\vec{u}$ is the velocity of the particle in m/s.

The magnetic flux density will be assumed to be as shown in Figure 1. The z component, B_{oz} , could be interpreted as the vertical component of the Earth's field and the x component, B_{ox} , as the horizontal component. In any event, $\vec{B}$ lies entirely within the xz plane. For an arbitrarily directed, linearly polarized electric field, $\vec{E}$, it is possible to reduce equation 1 under the assumption that the time variation is harmonic; ie, d/dt can be replaced by $i\omega = 2\pi i\nu$ where ω is the angular frequency and ν is the linear frequency. Then we have

$$mi\omega \vec{u} = q\vec{E} + q(\vec{u} \times \vec{B}). \quad (2)$$

There are several ways of proceeding from this point. One is to expand equation 2 in terms of the vector components and use Maxwell's equations to form the dielectric tensor:

$$\frac{1}{\mu} (\nabla \times \vec{B}) = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}, \quad (3)$$

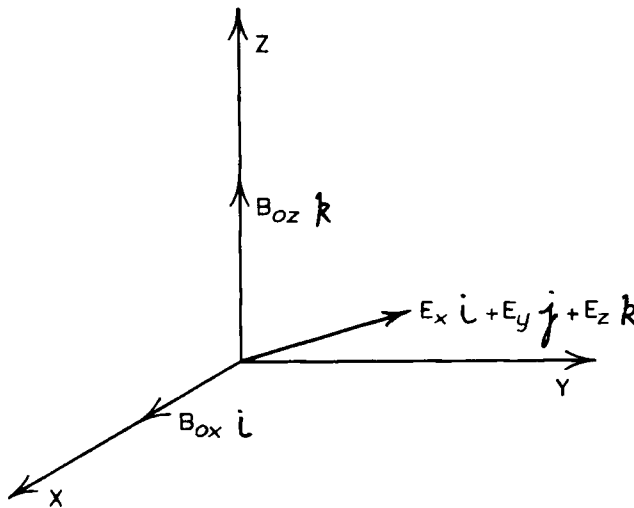


Fig. 1. Orientation of the magnetostatic (B) and periodic electric (E) field vectors.

where μ is the permeability, equal to $4\pi \times 10^{-7}$ Henrys/m, ϵ is the electric permittivity, in farads/m, and $\vec{J}$ is the current density, in amperes/m². If in unit volume there are N ions, each carrying the same charge, equation 3 can be rewritten:

$$\begin{aligned}\nabla \times \vec{B} &= \mu N q \vec{u} + i\omega\mu\epsilon \vec{E} \\ &= \left(\frac{Nq\vec{u}}{i\omega\epsilon} + 1 \right) i\omega\mu\epsilon \vec{E}\end{aligned}\quad (4)$$

$$= \bar{\kappa} i\omega\mu\epsilon \vec{E} \quad (5)$$

where $\bar{\kappa}$ is the relative dielectric tensor:

$$\bar{\kappa} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix} . \quad (6)$$

The elements κ_{ij} of this tensor, formed by enumerating the components of $\vec{u}$ in equation 4, are listed in Appendix A.

Note that terms of the form $\omega^2 - \omega_B^2$ appear in the denominators of the κ_{ij} . The defining equation for the cyclotron resonance gyrofrequency is

$$\omega_B = \frac{q}{m} \left(B_{oz}^2 + B_{ox}^2 \right)^{1/2} = \frac{q}{m} B_r , \quad (7)$$

where B_r is the resultant (magnetostatic) flux density.

Another way of proceeding from equation 2 is to express the velocity as

$$\vec{u} = \vec{u}_o + \vec{u}_e e^{i\omega t} , \quad (8)$$

where $\vec{u}_o$ is the noncyclic component of $\vec{u}$ and $\vec{u}_e$ is the cyclic component of $\vec{u}$.

Substituting equation 8 into 2 yields the steady-state Lorentz acceleration

$$\frac{d\vec{u}}{dt} = \frac{q}{m} (\vec{u}_o \times \vec{B}) , \quad (9)$$

and also

$$\frac{q}{m} \vec{E} = (\vec{\omega}_B \times \vec{u}_e) + i\omega \vec{u}_e , \quad (10)$$

where

$$\vec{\omega}_B = \frac{q}{m} \left(B_{oz}^2 + B_{ox}^2 \right)^{1/2} \vec{a}_B , \quad (11)$$

and $\vec{a}_B$ is the unit vector in the direction of $\vec{B}$.

Note that $\vec{\omega}_B$ is the vector angular frequency pointing in the direction of $\vec{B}$ and carrying the magnitude of cyclotron resonance frequency as in equation 7.

From equation 10, one can obtain the components of particle velocity parallel and perpendicular to $\vec{\omega}_B$ as well as expressions for left and right rotating velocity vectors. The details are given in Appendix B. One obtains

$$\vec{u}_{\parallel} = -\frac{iq}{m\omega} \vec{E}_{\parallel} \quad (12)$$

$$\vec{u}_{\perp} = \frac{q}{m} \left(\frac{i\omega - \vec{\omega}_B \times}{\omega^2 - \omega_B^2} \right) \vec{E}_{\perp} , \quad (13)$$

where $\vec{E}_{\parallel}$ is the electric field vector parallel to $\vec{\omega}_B$, such that

$$\vec{E}_{\parallel} = \frac{\vec{E} \cdot \vec{\omega}_B}{\omega_B} \vec{a}_B , \quad (14)$$

and $\vec{E}_{\perp}$ is the E-vector normal to $\vec{\omega}_B$,

$$\vec{E}_{\perp} = \frac{\omega_B \times \vec{E}}{\omega_B} . \quad (15)$$

One can define left and right rotating components of the electric field

$$\vec{E}_L = \frac{1}{2} \left[\vec{E}_{\perp} + i \left(\frac{\vec{\omega}_B \times \vec{E}_{\perp}}{\omega_B} \right) \right] \quad (16)$$

and

$$\vec{E}_R = \frac{1}{2} \left[\vec{E}_{\perp} - i \left(\frac{\vec{\omega}_B \times \vec{E}_{\perp}}{\omega_B} \right) \right] \quad (17)$$

This enables us to define a corresponding pair of rotating velocity vectors:

$$\vec{u}_L = \frac{q}{m} \left(\frac{i}{\omega_B - \omega} \right) \vec{E}_L \quad (18)$$

$$\vec{u}_R = -\frac{q}{m} \left(\frac{i}{\omega_B + \omega} \right) \vec{E}_R \quad (19)$$

The first of these, $\vec{u}_L$, is a velocity vector attached to the particle, rotating to the left, looking along $\vec{\omega}_B$, and $\vec{u}_R$ is the corresponding vector rotating to the right (see Fig. 2).

The cyclotron resonance frequency, ω_B , appears explicitly in the denominators, as in the relative dielectric tensor. For a positively charged ion there is a resonance in $\vec{u}_L$ when $\omega \rightarrow \omega_B$. However, $\vec{u}_R$ also shows a resonance for negatively charged particles, inasmuch as ω_B is negative in this case:

$$\vec{u}_R = -\frac{|q|}{m} \frac{i}{|\omega_B| - \omega} \vec{E}_R. \quad (20)$$

The overall dependence of $\vec{u}$ on $\vec{E}$ will be characterized in terms of the mobility tensor, the elements of which can be found in equations 12, 18, and 19:

$$\begin{bmatrix} u_L \\ u_R \\ u_{\parallel} \end{bmatrix} = i \frac{q}{m} \begin{bmatrix} \frac{1}{\omega_B - \omega} & 0 & 0 \\ 0 & \frac{-1}{\omega_B + \omega} & 0 \\ 0 & 0 & \frac{-1}{\omega} \end{bmatrix} \begin{bmatrix} E_L \\ E_R \\ E_{\parallel} \end{bmatrix}. \quad (21)$$

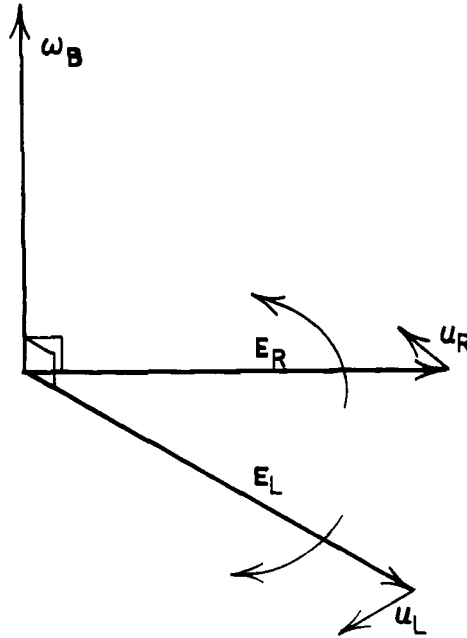


Fig. 2. Relationship of the circularly rotating electric (E) and velocity (u) vectors of the resonance angular frequency vector, $\vec{\omega}_B$.

DISCUSSION

The aforementioned model suggests that biostructures should exhibit responses at specific ionic cyclotron resonance frequencies. These frequencies can be applied either as periodic electric or periodic magnetic fields. We have deliberately chosen the former case, as stated in equation 1, anticipating that the introduction of a time-varying B-field requires a more extensive analysis than has been presented above (except for simple solenoidal B-fields). However, it is possible to approach this question by asking how the cyclotron resonance condition in equation 7 would vary with the application of a superposed periodic B-field. Let us therefore add to the geomagnetic field an additional harmonic B-field directed along the x axis:

$$\vec{B}_{xac} = B_{xac} \cos \omega t \hat{i}. \quad (22)$$

The effects caused by this superposed field will vary depending on its magnitude and frequency. There will be a quasistatic change in ω_B providing that the period in equation 22 is longer than the time it takes a resonant ion to describe one full loop in its path. In other words, if the applied frequency is kept small, such that $\omega \ll \omega_B$, the gyrofrequency will change according to the expression

$$\omega'_B = \frac{q}{m} [B_{oz}^2 + (B_{ox} + B_{ac} \cos \omega t)^2]^{1/2}, \quad (23)$$

where ω'_B is the slowly changing resonance frequency of an ion, this new frequency changing both in magnitude and direction. We can rewrite equation 23 in terms of the static gyrofrequency:

$$\omega'_B = \omega_B \left[1 + \frac{2B_{ox}B_{xac} \cos \omega t}{B_r^2} + \frac{B_{xac}^2}{B_r^2} \cos^2 \omega t \right]^{1/2}. \quad (24)$$

To study the behavior of equation 24, let us suppose that $B_{oz} = B_{ox}$ such that $\vec{B}_r$ is inclined at 45° to $\vec{B}_{xac}$ and furthermore that $B_{xac} = 10 B_{ox}$. If we ask how close ω'_B is to ω_B over one complete period, we find a rather wide variation (Fig. 3A). One interesting point is that when $B_{xac} \gg B_{ox}$, the resonance frequency is never less than the static value.

On the other hand, if the applied field is less than the static field, a quite different picture results. For example, let $B_{xac} = 0.1 B_{ox}$. Figure 3B shows that ω'_B and ω_B are within 5% of one another over one period. In Figure 4 we show two other cases, first for $B_{xac} = B_{ox}$ and, second, for $B_{xac} = 1.5 B_{ox}$. It is apparent that a wide range of resonant conditions are possible depending on the applied magnitude B_{xac} and frequency ω . Again, we caution the reader that the concept of a quasistatically varying ω'_B is valid only when $\omega'_B \ll \omega_B$.

Even if we do not admit such a quasistatic variation in ω_B , it seems clear that it is nevertheless interesting to consider what effects might occur when $\omega = \omega_B$. For example, we note that the times for which $\omega'_B = \omega_B$ are given by

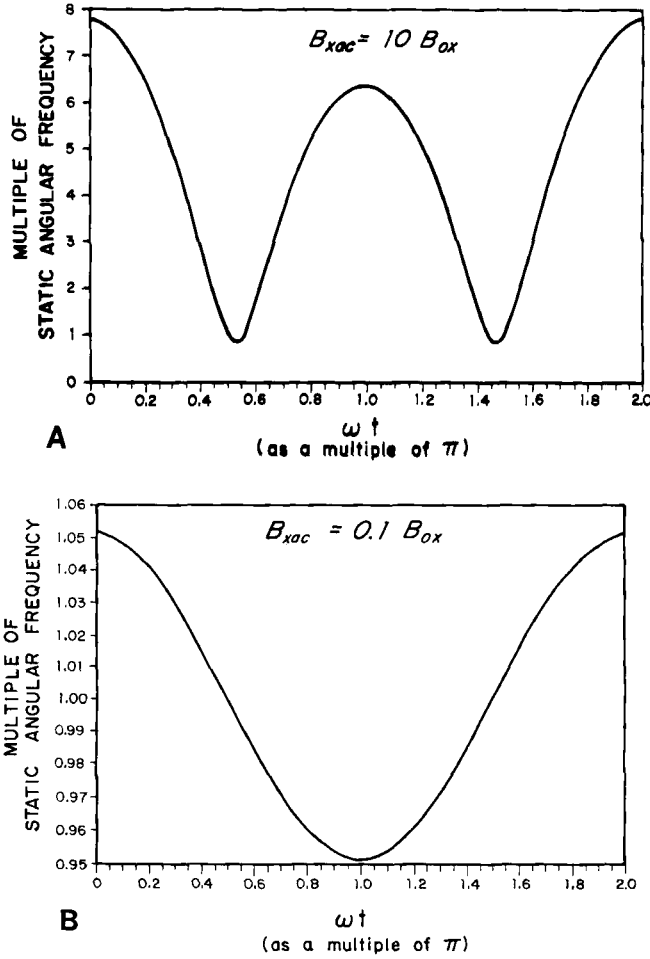


Fig. 3. The system cyclotron resonance frequency ω'_B as a multiple of the static cyclotron resonance frequency, ω_B . For 3A, it is assumed that $B_{oz} = B_{ox}$ (ie, the geomagnetic vector has a declination angle of 45°) and that $B_{xac} = 10 B_{ox}$ (ie, the applied alternating field is large compared to B_{ox}). For 3B, the same conditions hold except that $B_{xac} = 0.1 B_{ox}$ so that the applied field is small compared to B_{ox} . In both cases, B_{xac} is applied along the x axis parallel to B_{ox} .

$$\omega t = (2n + 1) \frac{\pi}{2}, \quad n = 0, 1, 2, \dots \quad (25)$$

If, for a system of specific ions moving in helical paths at cyclotron resonance, then

$$\omega_B t_s = \frac{q}{m} B_r t_s = (2n + 1) \frac{\pi}{2} \quad (26)$$

or

$$t_s = \left(\frac{2n + 1}{2} \right) \frac{\pi m}{B_r q}, \quad (27)$$

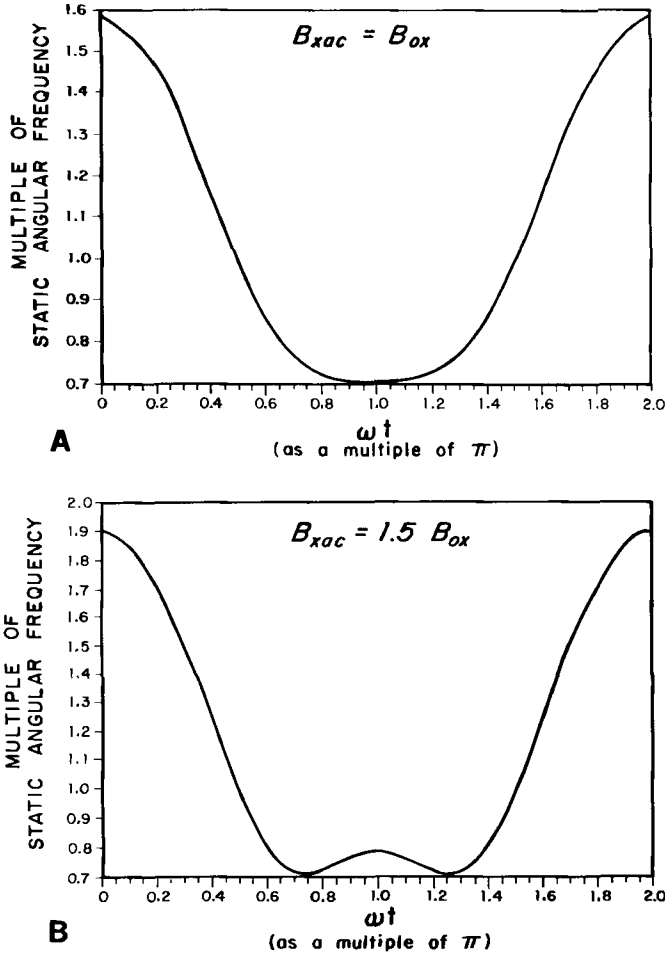


Fig. 4. ω_B as a multiple of ω_B . In 4A, the applied field $B_{xac} = B_{ox}$. In 4B, $B_{xac} = 1.5 B_{ox}$.

where t_s is the successive times when the cyclotron resonance condition for the system equals the static resonance condition. Note that the time interval between these successive events is constant:

$$\delta t_s = t_{n+1} - t_n = \frac{\pi}{B_r} \frac{m}{q} = \frac{\pi}{\omega_B}. \quad (28)$$

In Table 1 we calculate a number of these times for four ions of biological interest, for a magnetostatic field $B_r = 0.38$ gauss.

A separate question relates to the energy that may be extracted from applied periodic electric fields at frequencies close to resonance. Let us transform equation 21 back into Cartesian coordinates, first assuming that the horizontal component of the earth's field is vanishingly small ($B_{ox} = 0$). The velocity vector then becomes, neglecting damping:

TABLE 1. Calculated Times (in msec) When the Dynamic Cyclotron Resonance Equals the Static Cyclotron Resonance (for 0.38 gauss)

Ion	$t_{n=0}(10^{-3} \text{ s})$	$t_{n=1}(10^{-3} \text{ s})$	$\delta t(10^{-3} \text{ s})$
Li^+	3.0	8.9	5.9
Ca^{2+}	8.6	25.8	17.2
Na^+	9.9	29.6	19.7
K^+	16.8	50.4	33.6

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \frac{q}{m} \begin{bmatrix} \frac{i\omega}{\omega_B^2 - \omega^2} & \frac{\omega_B}{\omega_B^2 - \omega^2} & 0 \\ -\frac{\omega_B}{\omega_B^2 - \omega^2} & \frac{i\omega}{\omega_B^2 - \omega^2} & 0 \\ 0 & 0 & -\frac{i}{\omega} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \quad (29)$$

This allows us to express the kinetic energy of the ion as a function of the electric field

$$\text{KE} = \frac{m}{2} \left(\frac{q}{m} \right)^2 \left(\frac{E_x^2 + E_y^2}{\omega_B^2 - \omega^2} - \frac{E_z^2}{\omega^2} \right) \quad (30)$$

Assume that the electric field is applied normally to the earth's (vertical) magnetic field so that $E_z = 0$. As the applied frequency approaches ω_B , we can write

$$\omega_B^2 - \omega^2 = 2\omega_B \cdot \delta\omega, \quad (31)$$

where $\delta\omega = \omega_B - \omega$. In this case the kinetic energy reduces to:

$$\text{KE} = \frac{q^2}{4m} \cdot \frac{E_{\perp}^2}{\omega_B \cdot \delta\omega}. \quad (32)$$

Lacking precise information in the literature as to the potential distribution *within* membrane channels, the connection between the radius of a given helix and the particle energy is difficult to estimate. Nevertheless, we can point out that equation 32 implies that, for applied frequencies about 1 Hz removed from a static gyrofrequency of 15 Hz, the application of an electric field of magnitude ~ 0.1 V/m would result in an energy transfer of about 10 eV, well in excess of kT. However, this calculation suffers from not knowing the degree of damping. One must first reduce the velocity in equation 29 by the appropriate damping factor before coming to a firm estimate of the useful energy that can be extracted from the applied field. Indeed, it is likely that an estimate of the damping factor is critical to the further development of this model.

CONCLUSIONS

An effort has been made in this paper to provide the relevant groundwork that may serve to relate cyclotron resonance theory to biosystems, particularly those enjoying helical symmetry. The basic structure of transmembrane protein channels is currently under active study. It is by no means clear that such channels either take the shape of or support helical pathways. Indeed, the simplest model is a straight cylinder with a bore large enough to accommodate the carrier. However, there is sufficient evidence in the literature to indicate that channel walls are composed of α -helical subunits, suggesting the possibility of an intrinsic helicity, either as a structural feature or in terms of the electric potential. One example incorporating helical subunits is found in the structure of the photon-driven proton pump, bacteriorhodopsin [Alberts et al, 1983], a protein that spans the membrane as seven twisted α -helices. Still another is the acetylcholine receptor, a complex of three to five α -helical transmembrane units [Popot and Changeux, 1984]. Finally, the gramicidin A transmembrane cation channel has a well characterized α -helical structure [Weinstein et al, 1979]. The model proposed by Liboff [1985] allows one to design various experiments that could be used to test its validity in terms of such classical concepts as the motion of charged particles in static and periodic fields. However, certain difficulties result, not the least of which is that the analysis of the motion of ions in time-varying magnetic fields is somewhat more complicated than for time-varying electric fields. At frequencies $\omega \ll \omega_B$, the cyclotron resonance frequency for a biosystem will vary depending on the relative amplitudes of the geomagnetic and applied fields. Judging from Figures 3 and 4, it might be possible to adjust these parameters for optimal response. Estimates of possible modulation times for cyclotron resonance conditions appear to be of the order of tens of msec.

One interesting result is that, because of the circular decomposition of the electric field into left and right components, it would appear that oppositely charged ions can be accelerated in the same direction through trans-membrane helical channels.

Realistic estimates of the energy transferred from the periodic field to the ion are not yet possible inasmuch as loss mechanisms have not been included in our analysis. Nevertheless, a reasonable estimate at relatively modest field strengths (< 1 V/m) indicates that energy transfers in excess of kT should readily occur for helically constrained membrane ions at geomagnetic field strengths. It is worth speculating to suggest that some of the wide variations in biomagnetic experimental results in the literature, even for seemingly similar experiments, can be accounted for if geomagnetic cyclotron resonance does indeed play a key biologic role. Virtually none of the experiments in the past enjoyed a sufficiently controlled or measured magnetic environment. Judging from the introductory analysis in this paper, the interplay of parameters is sufficiently complex to preclude accidentally reproducible experiments.

Finally, one can compare cyclotron resonance coupling with earlier models of interaction between time-varying fields and cells. For experiments in which cell response was studied as a function of eddy-current density, little or no variation with intensity was observed [Sisken et al, 1984; Liboff et al, 1984]. One interpretation of these results is that time-varying fields may have their focus of interaction within the cell [Liboff and Homer, 1983] rather than at the surface. However, this would imply that there are biologically meaningful levels of intracellular current density much smaller than the more reasonable $1 \mu\text{A}/\text{cm}^2$ originally calculated by McLeod et al

[1983] for the case of surface current density. There is the possibility that cyclotron resonance serves to pump intracellular regions such as channels by increasing the local effective conductivity.

REFERENCES

- Alberts B, Bray C, Lewis J, Raff M, Roberts K, Watson JD (1983): "Molecular Biology for the Cell" New York: Garland Publ Co, p 272.
- Blackman CF, Benane SG, Rabinowitz JR, House DE, Joines WT (1985): A role for the magnetic field in the radiation-induced efflux of calcium ions from brain tissue in vitro. *Bioelectromagnetics* 6:327-337.
- Liboff AR (1985): Cyclotron resonance in membrane transport. In Chiabrera A, Nicolini C, Schwan HP (eds): "Interactions Between Electromagnetic Fields and Cells." London: Plenum, pp 281-296.
- Liboff AR, Homer L (1983): Eddy-current effects on DNA synthesis at geomagnetic intensities. In "Proc Bioelectromagnetics Society Fifth Annual Meeting, Boulder, CO." p 3a.
- Liboff AR, Williams T Jr, Strong DM, Wistar R Jr (1984): Time-varying magnetic fields: Effect on DNA synthesis. *Science* 223:818-820.
- McLeod BR, Pilla AA, Sampsel MW (1983): Electromagnetic fields induced by Helmholtz aiding coils inside saline filled boundaries. *Bioelectromagnetics* 4:357-370.
- Popot J, Changeux J (1984): Nicotinic receptor of acetylcholine: Structure of an oligomeric integral membrane protein. *Physiol Rev* 64:1162-1239.
- Sisken BF, McLeod B, Pilla AA (1984): PEMF, direct current and neuronal regeneration: Effect of field geometry and current density. *J Bioelectricity* 3:81-101.
- Weinstein S, Wallace BA, Plant ER, Morrow JS, Veatch W (1979): Conformation of gramicidin A channel in phospholipid vesicles: A ^{13}C and ^{19}F nuclear magnetic resonance study. *Proc Natl Acad Sci USA* 76:4230-4234.

APPENDIX A

The relative dielectric tensor has the general form:

$$\bar{\bar{\kappa}} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix}.$$

where, for the present case

$$\kappa_{11} = 1 - \frac{\omega_p^2 (\omega^2 - \omega_{BX}^2)}{\omega^2 (\omega^2 - \omega_B^2)}$$

$$\kappa_{12} = -\kappa_{21} = \frac{i\omega_p^2 \omega_{BZ}}{\omega(\omega^2 - \omega_B^2)}$$

$$\kappa_{23} = -\kappa_{32} = \frac{i\omega_p^2 \omega_{BX}}{\omega(\omega^2 - \omega_B^2)}$$

$$\kappa_{13} = \kappa_{31} = \frac{\omega_p^2 \omega_{BX} \omega_{BZ}}{\omega^2 (\omega^2 - \omega_B^2)}$$

$$\kappa_{22} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_B^2}$$

$$\kappa_{33} = 1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2 \omega_{BX}^2}{\omega^2 (\omega^2 - \omega_B^2)},$$

and

$$\omega_p^2 = \frac{Nq^2}{\omega m}$$

$$\omega_{BX} = \frac{q}{m} B_{ox}$$

$$\omega_{BZ} = \frac{q}{m} B_{oz}$$

$$\omega_B = \frac{q}{m} (B_{oz}^2 + B_{ox}^2)^{1/2}.$$

APPENDIX B

The cyclic position of the Lorentz force equations is given by:

$$\frac{q}{m} \vec{E} = (i\omega + \vec{\omega}_B \times) \vec{u}_e. \quad (B1)$$

If one multiplies by the conjugate vector operator $(-i\omega + \vec{\omega}_B \times)$, the result is

$$\frac{q}{m} (-i\omega + \vec{\omega}_B \times) \vec{E} = (-i\omega + \vec{\omega}_B \times) (i\omega + \vec{\omega}_B \times) \vec{u}_e, \quad (B2)$$

or

$$\frac{q}{m} (\omega_B \times \vec{E}) - \frac{iq}{m} \omega \vec{E} = (\omega^2 - \omega_B^2) \vec{u}_e + (\vec{\omega}_B \cdot \vec{u}_e) \vec{\omega}_B. \quad (B3)$$

$$\frac{q}{m} (\vec{\omega}_B \times \vec{E}) \cdot \vec{\omega}_B - \frac{iq\omega}{m} (\vec{E} \cdot \vec{\omega}_B) = \quad (B4)$$

$$(\omega^2 - \omega_B^2) \vec{u}_e \cdot \vec{\omega}_B + (\vec{\omega}_B \cdot \vec{u}_e) \vec{\omega}_B \cdot \vec{\omega}_B,$$

or

$$-\frac{iq\omega}{m} \vec{E}_{\parallel} = (\omega^2 - \omega_B^2 + \omega_B^2) \vec{u}_{\parallel} \quad (B5)$$

$$\vec{u}_{\parallel} = -\frac{iq}{\omega m} \vec{E}_{\parallel}. \quad (B6)$$

Similarly, the component of velocity perpendicular to $\vec{\omega}_B$ is found by forming the vector product of $\vec{\omega}_B$ with equation B3. Thus

$$\begin{aligned} \frac{q}{m} (\vec{\omega}_B \times \vec{\omega}_B \times \vec{E}) - \frac{iq\omega}{m} (\vec{\omega}_B \times \vec{E}) = \\ (\omega^2 - \omega_B^2) (\vec{\omega}_B \times \vec{u}_\perp) + (\vec{\omega}_B \cdot \vec{u}_\perp) (\vec{\omega}_B \times \vec{\omega}_B) \end{aligned} \quad (B7)$$

or

$$\frac{q}{m} (\vec{\omega}_B \times \vec{E}_\perp) - \frac{iq}{m} \omega \vec{E}_\perp = (\omega^2 - \omega_B^2) \vec{u}_\perp, \quad (B8)$$

resulting in

$$\vec{u}_\perp = \frac{q}{m} \left[\frac{i\omega \vec{E}_\perp - (\vec{\omega}_B \times \vec{E}_\perp)}{\omega^2 - \omega_B^2} \right]. \quad (B9)$$